Cosmologies in Horndeski's second-order vector-tensor theory

John D. Barrow,[a] **Mikjel Thorsrud**[b] **and Kei Yamamoto**[a,b]

[a]*DAMTP, University of Cambridge,*
Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

[b]*Institute of Theoretical Astrophysics, University of Oslo,*
P.O. Box 1029 Blindern, N-0315 Oslo, Norway

E-mail: J.D.Barrow@damtp.cam.ac.uk, mikjel.thorsrud@astro.uio.no,
K.Yamamoto@damtp.cam.ac.uk

ABSTRACT: Horndeski derived a most general vector-tensor theory in which the vector field respects the gauge symmetry and the resulting dynamical equations are of second order. The action contains only one free parameter, λ, that determines the strength of the non-minimal coupling between the gauge field and gravity. We investigate the cosmological consequences of this action and discuss observational constraints. For $\lambda < 0$ we identify singularities where the deceleration parameter diverges within a finite proper time. This effectively rules out any sensible cosmological application of the theory for a negative non-minimal coupling. We also find a range of parameter that gives a viable cosmology and study the phenomenology for this case. Observational constraints on the value of the coupling are rather weak since the interaction is higher-order in space-time curvature.

Contents

1 Introduction

In the past few decades, modifying the Einstein's theory of gravitation has been an active area of research [1], driven chiefly by the search for different varieties of inflation, the desire of some to explain flat galaxy rotation curves without dark matter [2], and the challenge of explaining why the expansion of the universe started to accelerate at late times. It is also natural to question the validity of general relativity, not least because of its ultra-violet behaviour which does not give a well-defined quantum field theory. There is a growing prospect of testing any such deviations from general relativity in very strong gravity fields by searching for the signatures of gravitational waves created by high-energy astrophysical phenomena, such as black hole mergers, or by scrutinising detailed observations of the microwave background anisotropy and statistics in the light of particular theories of inflation. While there are a plethora of inflationary models, it is difficult to modify general relativity without spoiling its appealing features and typical modifications end up introducing new

scalar degrees of freedom, as was the case in the case of Brans-Dicke gravity [3] and its scalar-tensor generalisations [4–6], or $f(R)$ lagrangian theories of gravity [7–11].

Maxwell's classical theory of electromagnetism has also been extremely well tested and there are already strict constraints on potential modifications such as a non-zero photon mass [12–15] or varying fine structure constant [16–24]. However, there has been renewed interest in modified electromagnetism in cosmology and there have been attempts to incorporate its effects into the dynamics of early universe, particularly during inflation, or to provide an explanation for cosmological magnetic fields [25–40]. Simple vector fields themselves are known to have difficulties in producing inflation. The time variation of the vector field is governed by the covariant time derivative of the field. Since the Christoffel symbols for an expanding cosmological model are of order the Hubble expansion rate it is not possible for the vector field to satisfy a slow-roll condition in the way that a scalar field can [41]. However Einstein-aether theories offer an alternative that permits inflation [1, 24, 42–51]. In addition, although it was noted that a non-minimal coupling to the space-time curvature could drive accelerated expansion [52–55], these scenarios suffer from various instabilities created by additional degrees of freedom arising from the lost gauge symmetry [56–58]. A different type of extension of the Maxwell case is provided by the extension to Yang-Mills fields where there can be chaotic behaviour and arbitrarily low levels of anisotropy [59–61].

Recently, it has been mentioned that a vector-tensor theory, first proposed by Horndeski in 1976 [62], could lead to an instability of conventional inflationary universe through the non-minimal coupling between the vector field and gravity [63]. The action was derived by demanding second-order dynamical equations that reduce to Maxwell's equations when evaluated on a Minkowski background and conservation of the $U(1)$ current. These requirements result in only one additional term in the Lagrangian, and therefore a single free coupling parameter. Later, it was noticed that this theory falls into a special class of Kaluza-Klein reductions from higher-dimensional Lovelock invariants [64–66]. In contrast to the Horndeski scalar field theory [67], which has been discussed in attempts to construct the most general viable scalar-tensor theory recently, [68–73], except for a brief examination of the static electromagnetism arising from this action [74], it appears to have escaped attention. Apart from being briefly mentioned in [63], its cosmological consequences have not been studied.

In this paper, we will investigate the simplest cosmological model, which is well understood in the minimally coupled case of a Maxwell electromagnetic field [75, 76], containing a perfect fluid and a vector field whose dynamics are described by the Horndeski Lagrangian. We find the following results;

1. The instability found in [63] for negative values of the coupling constant persists in the nonlinear regime and the universe eventually hits a singularity;

2. For a positive coupling constant, the electric field can still be amplified during the radiation-dominated era while giving a viable cosmology subject to some constraints on the allowed expansion rate changes at the epoch of primordial nucleosynthesis.

The first result effectively rules out any interesting cosmological application of this theory with a negative coupling. For a positive coupling constant, the modification is rather tame and an enormous value of the coupling in units of the Planck mass is allowed because of the higher-order nature of the modified term. However, the dynamics is of a phenomenological interest.

The article is organised as follows. In the next section, the theory is introduced and the modified Einstein-Maxwell equations are presented. Section 3 is the main part of the article where the dynamics of purely electric component are studied in an axisymmetric Bianchi type I universe. In section 4, we repeat the previous analysis for magnetic component. Section 5 discusses observational constraints and in section 6 we summarise our principal results.

2   Horndeski's second-order vector-tensor theory

In 1976, Horndeski showed that the general Lagrangian that can be constructed from a metric g_{ab} and a vector field A_a in four-dimensional space-time that satisfies the following conditions [62]:

1. the field equations contain at most second-order derivatives of g_{ab} and A_a (and do contain a second-order term);

2. the dynamical equations for A_a respect charge conservation i.e. $\nabla_a(\partial \mathcal{L}/\partial A_a) = 0$;

3. the dynamical equations for A_a reduce to Maxwell's equations when evaluated on Minkowski space-time;

takes the following form:

$$\mathcal{L} = \frac{M_{pl}^2}{2}\sqrt{-g}R - \frac{1}{4}\sqrt{-g}F_{ab}F^{ab} + \mathcal{L}_H, \tag{2.1}$$

where M_{pl} is the reduced Planck mass, R is the Ricci scalar and $F_{ab} = \partial_a A_b - \partial_b A_a$ is the Faraday tensor. The last term is Horndeski's modification which can be expressed in several different ways as

$$\mathcal{L}_H = -\frac{3\lambda}{2M_{pl}^2}\sqrt{-g}\delta_{efkl}^{abcd}F_{ab}F^{ef}R_{cd}{}^{kl} \tag{2.2}$$

$$= \frac{\lambda}{4M_{pl}^2}\sqrt{-g}F_{ab}F^{cd} * R *^{ab}{}_{cd} \tag{2.3}$$

$$= -\frac{\lambda}{4M_{pl}^2}\sqrt{-g}\left(RF_{ab}F^{ab} - 4R_{ab}F^{ac}F^b{}_c + R_{abcd}F^{ab}F^{cd}\right). \tag{2.4}$$

The dimensionless non-minimal coupling constant λ is the only parameter of the theory. Our aim is to investigate the cosmological consequences with an arbitrary value of λ and to determine the parameter range yielding viable phenomenology. The other terms in the

Lagrangian (2.1) are normalized so that it reduces to the Einstein-Maxwell theory when $\lambda = 0$. Using the Levi-Civita tensor η_{abcd}, we defined the generalised Kronecker's delta by

$$\delta^{abcd}_{efkl} = \delta^a_{[e}\delta^b_f\delta^c_k\delta^d_{l]} = -\frac{1}{24}\eta^{abcd}\eta_{efkl}$$

and the double dual of Riemann by

$$*R*^{ab}_{\ \ cd} = \frac{1}{4}\eta^{abef}\eta_{cdkl}R_{ef}^{\ \ kl}.$$

$\mathcal{L}_H$ was later identified with the Lagrangian obtained by Kaluza-Klein reduction from the five-dimensional Lovelock invariant

$$K = \tilde{R}_{abcd}\tilde{R}^{abcd} - 4\tilde{R}_{ab}\tilde{R}^{ab} + \tilde{R}^2$$

where $\tilde{R}_{abcd}$ is the Riemann tensor in five dimensions [64].

In this paper we use the sign conventions of [77] for the metric, Ricci and Riemann tensors which are different from those adopted in the previous studies of this model [62, 74]. The dynamical equations derived from this Lagrangian are given as follows:

Variation with respect to g_{ij}

$$M_{pl}^2 G_{ij} = F_{ia}F_j^{\ a} - \frac{1}{4}F_{ab}F^{ab}g_{ij} + \tau_{ij}, \tag{2.5}$$

where

$$\tau_{ij} = \frac{\lambda}{M_{pl}^2}\left(\nabla_a * F_{ib}\nabla^b * F^a_{\ j} + F^{ab}F^c_{\ a} * R*_{ibjc}\right), \tag{2.6}$$

Variation with respect to A_i

$$\nabla_a F^{ia} - \frac{\lambda}{M_{pl}^2}\nabla_a F_{bc} * R*^{iabc} = 0. \tag{2.7}$$

We define the dual Faraday tensor as usual:

$$*F_{ab} = \frac{1}{2}\eta_{abcd}F^{cd}.$$

In ref.[63], it was observed that (2.7) evaluated on a Friedmann-Lemaître-Robertson-Walker (FLRW) background could lead to an instability of the vector field. Ignoring the spatial gradient term, the solution for the comoving electric field strength E evaluated on this background is given by

$$E = \frac{E_0}{a^2}\left(1 + 2\lambda\frac{H^2}{M_{pl}^2}\right)^{-1}, \tag{2.8}$$

where E_0 is an integration constant, a is the scale factor and H is the Hubble expansion rate. When λ is negative and $-2\lambda H^2 \gtrsim M_{pl}^2$, the energy density of the electric field can rapidly increase and eventually diverge, even when the expansion of the universe is accelerated. Our first goal is to take into account the back reaction of this growing vector field and examine the fate of the inflationary universe.

3 Dynamics of electric fields in axisymmetric Bianchi type I universes

In this section and the next, we set $M_{pl}^2 = 1$.

3.1 Electric fields in an anisotropic universe

In order to answer the question of back reaction, one needs to look at a fully non-linear system and solve both (2.5) and (2.7). The simplest generalisation of the FLRW universe that can accommodate a vector field is the axisymmetric Bianchi type I metric given by

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left[e^{-4\beta(t)} dx^2 + e^{2\beta(t)} \left(dy^2 + dz^2 \right) \right]. \tag{3.1}$$

This metric is spatially flat. Note that the (mean) Hubble and shear expansion rates are given by

$$H = \dot{\alpha}, \qquad \sigma = \dot{\beta}.$$

We consider a homogeneous electric field along the x-direction, which in the gauge $A_0 = 0$ corresponds to the following coordinate basis components for the vector potential:

$$A_\mu = (0, A(t), 0, 0).$$

The electric field strength seen by an observer moving with four-velocity $u^\mu = (1, 0, 0, 0)$ is given by

$$E(t) = -\dot{A} e^{-\alpha + 2\beta},$$

where dots denote derivatives with respect to the comoving proper time t. We also include a perfect fluid with the equation of state $p = (\gamma - 1)\rho$ and constant γ. Hence, (2.5) yields the following:

$$H^2 - \sigma^2 = \lambda (H + \sigma)^2 E^2 + \frac{1}{6} E^2 + \frac{1}{3} \rho, \tag{3.2}$$

$$\dot{H} + H^2 = -2\sigma^2 - \frac{1}{6} E^2 - \frac{1}{6}(3\gamma - 2)\rho - \frac{\lambda}{6} (H + \sigma)(H + 7\sigma) E^2 \tag{3.3}$$

$$+ \frac{\lambda E^2}{6} \left[\lambda (H + \sigma)^2 E^2 + \frac{1}{2} E^2 - (\gamma - 1)\rho + 4 (H + \sigma) \frac{\dot{E}}{E} \right],$$

$$\dot{\sigma} = -3H\sigma + \frac{1}{3} E^2 + \frac{\lambda}{6} (H + \sigma)(H + 7\sigma) E^2 \tag{3.4}$$

$$- \frac{\lambda E^2}{6} \left[\lambda (H + \sigma)^2 E^2 + \frac{1}{2} E^2 - (\gamma - 1)\rho + 4 (H + \sigma) \frac{\dot{E}}{E} \right].$$

We have already reshuffled the Einstein equations to put them into a convenient form. Eqs.(3.2) and (3.3) correspond to the Friedmann and Raychaudhuri equations, respectively. Eq.(2.7) can be written as

$$\dot{E} = -\frac{2 (H + \sigma)}{1 + 2\lambda (H + \sigma)^2} \left[1 + 2\lambda (H + \sigma)^2 + 2\lambda \left(\dot{H} + \dot{\sigma} \right) \right] E. \tag{3.5}$$

The fluid obeys the usual adiabatic decay law:

$$\dot{\rho} = -3\gamma \dot{\alpha} \rho. \tag{3.6}$$

Equations (3.2) - (3.6) form a closed set of non-linear ordinary differential equations. The Friedmann equation (3.2) measures the dynamical significance of each matter component.

To gain an insight into the effect of the Horndeski's extra non-minimal coupling, let us assume that Horndeski's modification term is dominant, that is

$$\left| \lambda \left(H + \sigma \right)^2 E^2 \right| \gg E^2, \rho.$$

The Friedmann equation (3.2) becomes

$$H^2 - \sigma^2 \sim \lambda \left(H + \sigma \right)^2 E^2, \tag{3.7}$$

which means that the universe must be strongly anisotropic when $\lambda < 0$. Using the same approximation, (3.5) reduces to

$$\frac{\dot{E}}{E} \sim -\frac{1 - 2\lambda \sigma \left(H + \sigma \right)}{\lambda \left(H + \sigma \right)}.$$

Now, (3.3) becomes

$$\dot{H} + H^2 \sim -2\sigma^2,$$

which is equivalent to the usual Einstein equation in an empty Bianchi I universe dominated by the shear because the back-reaction of the Horndeski term exactly cancels out at leading order. The same is true in the shear evolution equation, as (3.4) yields

$$\dot{\sigma} \sim -3H\sigma,$$

so that the universe isotropises in the same way as a (flat) universe containing only perfect fluids. We conclude that the Horndeski modification $\mathcal{L}_H$ is fairly innocuous despite the formidable appearance of its energy-momentum tensor. It should only be able to affect the evolution of the universe when its contribution is comparable to the conventional Maxwell term or the matter.

3.1.1 Occurrence of a finite-time singularity

The evolution equation for electric field (3.5) is rather similar to the linearized equation yielding the solution (2.8), although it is fully non-linear in the present setup. In fact, we can integrate (3.5) analytically and obtain

$$E = \frac{E_0}{e^{2(\alpha+\beta)}} \frac{1}{1 + 2\lambda \left(H + \sigma \right)^2}. \tag{3.8}$$

This is essentially the same as the solution in FLRW background (2.8). Unless there is a mechanism within (3.2) - (3.6) that prevents $(H + \sigma)^2$ from reaching $-1/2\lambda$, there should be a range of initial conditions for which the system eventually hits a singularity. Since the system comes close to this singularity precisely when Horndeski's modified term becomes comparable to the Maxwell term

$$\left| \lambda \left(H + \sigma \right)^2 E^2 \right| \sim E^2,$$

we may see unusual dynamical behaviours in this regime.

Let us see what happens to the evolution of the spatial geometry when the system approaches the singularity. For this purpose, it is useful to write down the evolution equation for $H + \sigma$, which we can cast into the following form;

$$\dot{H} + \dot{\sigma} = -\frac{1}{2}\left[3\left(H + \sigma\right)^2 + (\gamma - 1)\rho\right] + \frac{1 + 2\lambda\left(H + \sigma\right)^2}{4}E^2. \tag{3.9}$$

It is immediately clear that the right-hand side is negative definite when $\gamma \geq 1$ and $\lambda < 0$ for initial conditions satisfying

$$\left(H + \sigma\right)^2 > -\frac{1}{2\lambda}. \tag{3.10}$$

Therefore, the singularity is inevitable regardless of the initial conditions for $\gamma \geq 1$ as long as $H + \sigma > 0$ is sufficiently large initially. Although it does not apply to inflationary universes with $0 < \gamma \ll 1$, we already know $H + \sigma$ decreases monotonically while ρ dominates the evolution of the universe. Thus, if the condition (3.10) holds initially, E eventually grows and any matter domination, and hence inflation, comes to an end. Once the electric field starts to dominate the dynamics, the right-hand side of (3.9) is again negative definite and the singularity must be reached. While we are unable to eliminate the possibility that $\dot{H} + \dot{\sigma}$ turns to positive and the universe manages to avoid the singularity during a brief period of matter-electric equality, numerical calculations suggest otherwise (see figure 1).
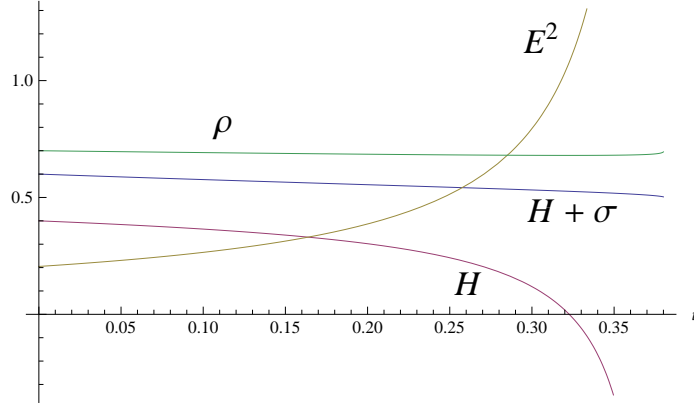

Figure 1. The approach towards the singularity for $\lambda = -2, \gamma = 0.1$ and $H + \sigma > 0$. The initial conditions are $\rho = 0.7, H = 0.4, \sigma = 0.2$. The universe is initially dominated by the matter with $p/\rho = -0.9$. There is no sign of avoiding the singularity located around $t \sim 0.38$. H becomes negative just before the singularity, which means the universe recollapses.

Since the instability condition (3.10) roughly corresponds to the one for Horndeski's term to have a significant effect compared to the Maxwell term in the Lagrangian, it effectively rules out any sensible cosmological application of the theory with a negative λ. We have not discussed the case of negative $H + \sigma$ since it describes either a collapsing universe or excessively anisotropic one. Close examination of (3.3) indicates recollapse or bounce right before hitting the singularity, both deriving from the violation of the weak

energy condition. These behaviours are also observed in our numerical solutions of the equations (see figure 2).
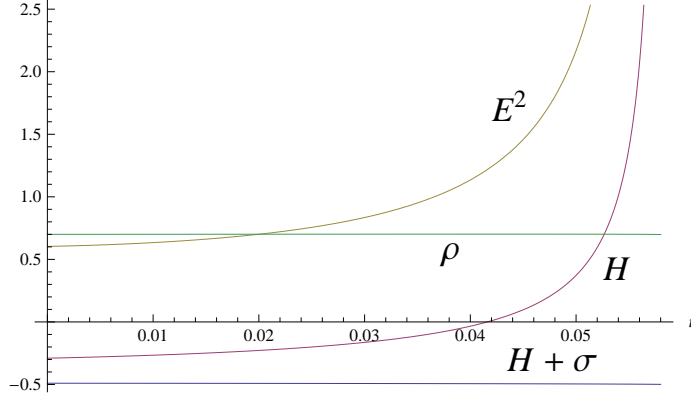

Figure 2. The approach towards the singularity for $\lambda = -2, \gamma = 0.1$ and $H + \sigma < 0$. The initial conditions are $\rho = 0.7, H = -0.29, \sigma = -0.2$. The universe is initially contracting, but eventually bounces and rapidly expands before it reaches the singularity around $t \sim 0.058$.

3.2 Expansion-normalised autonomous system

In the previous subsection, we saw that the theory with a negative coupling, λ, should lead to pathological dynamics when the Horndeski modification has an appreciable effect. On the other hand, when λ is positive there is no danger of a singularity. The energy density is positive definite and we expect some viable cosmological dynamics. In order to carry out a more systematic investigation, it is always useful to rewrite the equations in terms of density parameters defined for each matter component, including the Horndeski contribution. It also enables us to apply the conventional methods of dynamical systems analysis.

We introduce the following normalised variables:

$$\Sigma = \frac{\sigma}{H}, \qquad \Omega_M = \frac{E^2}{6H^2}, \qquad \Omega_H = 6\lambda \left(H + \sigma\right)^2 \Omega_M, \qquad \Omega_m = \frac{\rho}{3H^2},$$

where $\Omega_H \gtrless 0$ correspond to $\lambda \gtrless 0$ respectively. The normalised Friedmann equation

$$1 = \Sigma^2 + \Omega_H + \Omega_M + \Omega_m, \tag{3.11}$$

will be used as the standard measure of the dynamical significance of each component. In particular, when $\Omega_H > 0$, all the parameters are bounded by 1 so that their values have a clear physical interpretation. Note that $\Omega_i = \rho_i/3H^2$ for $i = (M, H)$, where ρ_M and ρ_H represent the minimal and non-minimal contribution to the vector field energy density,

respectively. From (3.3), we define the deceleration parameter as

$$
\begin{aligned}
q &= -\frac{\dot{H}}{H^2} - 1 \\
&= 2\Sigma^2 + \Omega_M + \frac{3\gamma - 2}{2}\Omega_m - \frac{(\Omega_M - \Omega_H)\,\Omega_H}{2\,(1+\Sigma)^2}\left(1 - \frac{3(\gamma-1)\Omega_m}{3\Omega_M + \Omega_H}\right) \\
&\quad + \frac{\Omega_H}{2\,(1+\Sigma)\,(3\Omega_M + \Omega_H)}\left(3\,(3 + 5\Sigma)\,\Omega_M - (1-\Sigma)\,\Omega_H\right).
\end{aligned}
\tag{3.12}
$$

Following the standard method [78], we switch the time coordinate from t to α. Using (3.11) and (3.12), we derive the following evolution equations for the expansion-normalised variables:

$$
\frac{d\Sigma}{d\alpha} = (q - 2)\,\Sigma - 2\Sigma^2 + \Omega_M - \frac{3\gamma - 2}{2}\Omega_m + q,
\tag{3.13}
$$

$$
\frac{d\Omega_M}{d\alpha} = 2\Omega_M\left[q + 1 - \frac{\Omega_H}{1+\Sigma}\left(1 - \frac{3(\gamma-1)\Omega_m}{3\Omega_M + \Omega_H}\right) - \frac{6\Omega_M - \Omega_H}{3\Omega_M + \Omega_H}\,(1+\Sigma)\right],
\tag{3.14}
$$

$$
\begin{aligned}
\frac{d\Omega_H}{d\alpha} = 2\Omega_H\Bigg[&q - 2\Sigma + \frac{\Omega_M - \Omega_H}{1+\Sigma} - \frac{6\Omega_M - \Omega_H}{3\Omega_M + \Omega_H}\,(1+\Sigma) \\
&- \frac{\Omega_m}{1+\Sigma}\left(\frac{3\gamma - 2}{2} - \frac{3\,(\gamma-1)\,\Omega_H}{3\Omega_M + \Omega_H}\right)\Bigg],
\end{aligned}
\tag{3.15}
$$

$$
\frac{d\Omega_m}{d\alpha} = (2q - 3\gamma + 2)\,\Omega_m.
\tag{3.16}
$$

These four equations are not independent since they are related by the first integral (3.11).

3.2.1 Fixed points in the dynamical system

We first classify the fixed points. Since the subsystem specified by $\Omega_H = 0$ is identical to the magnetic Bianchi type I discussed in [75, 76], we know there must be at least four fixed points of physical interest:

Flat Friedmann universe : F

$$
(\Sigma, \Omega_M, \Omega_H, \Omega_m) = (0, 0, 0, 1).
$$

Electric Bianchi type I : E

$$
(\Sigma, \Omega_M, \Omega_H, \Omega_m) = \left(\frac{3\gamma - 4}{4}, \frac{3}{16}\,(2 - \gamma)\,(3\gamma - 4), 0, \frac{3}{8}(4 - \gamma)\right).
$$

The existence condition is $\gamma > 4/3$.

Kasner solutions : $K_\pm$

$$
(\Sigma, \Omega_M, \Omega_H, \Omega_m) = (\pm 1, 0, 0, 0).
$$

In addition, there appears a fixed point describing a universe dominated by the Horndeski energy density:

Ω_H-dominated universe : H_E

$$(\Sigma, \Omega_M, \Omega_H, \Omega_m) = (0, 0, 1, 0) \,.$$

The deceleration parameter for this solution is $q = 0$, which is consistent with the analysis in section 3.1 where we showed that the back-reaction of the vector field exactly cancels out to leading order when the energy density is dominated by Horndeski's modification term.

There are subtleties regarding these fixed points. First of all, E and H_E do not represent physical space-times when $\lambda \neq 0$ since they imply $H = \sigma = 0$ and $(H + \sigma)^{-1} = 0$ respectively. It does not mean these fixed points are irrelevant in the dynamics, however, since they may be reached asymptotically from finite H and σ in the far past or future. We shall see an example of this in figure 7. Secondly, F and $K_{\pm}$ must be treated with care since some of the denominators appearing in the evolution equations (3.13)-(3.16) vanish on those fixed points. Again, it does not mean they are unphysical since they are well-behaved when appropriate limits are taken for the numerators. But the analysis requires evaluation of $0/0$, which implies the stabilities may depend on the way the fixed point is approached. It is a consequence of the fact that H is not decoupled from the normalised variables, and implicitly appears in the definition of Ω_H. While a fixed point in the expansion-normalised equations usually represents a self-similar solution that is invariant under a scale transformation, the dynamical effect of the Horndeski modification depends on the scale of the curvature, or the size of the universe. Therefore, it is not surprising to see the stability change depending on each orbit with its specific value of H. We will find this is indeed the case.

3.3 Dynamics around the matter-dominated solution

From physical point of view, by far the most interesting fixed point is F because it can be regarded as a model of the late-time evolution for the universe when $\gamma = 1$ (dust) or $\gamma = 4/3$ (radiation), and also a model of inflation when $\gamma < 2/3$. We have already mentioned the instability against perturbations of the electric field for $\lambda < 0$. The condition for the occurrence of a singularity (3.10) translates into

$$3\Omega_M + \Omega_H < 0 \tag{3.17}$$

in the new variables. Here, we shall see that this condition coincides with the instability condition for F and otherwise the dynamics is trivial. We also study the stability for positive λ and show that it depends on the value of H.

3.3.1 Linearisation

For the purpose of linearisation around F, it turns out to be convenient to eliminate Ω_m using the Friedmann equation (3.11) and rewrite the equations as

$$\frac{d\Sigma}{d\alpha} = f_\Sigma(\Sigma, \Omega_M, \Omega_H), \tag{3.18}$$

$$\frac{d\Omega_M}{d\alpha} = f_M(\Sigma, \Omega_M, \Omega_H), \tag{3.19}$$

$$\frac{d\Omega_H}{d\alpha} = f_H(\Sigma, \Omega_M, \Omega_H), \tag{3.20}$$

whose right-hand sides we avoid writing down explicitly as they are lengthy. While there are apparent $0/0$s in those equations when they are evaluated on F, they should be all well defined if an appropriate limit is taken along an arbitrary reference orbit. To proceed, we evaluate the functions $f_i, i = \Sigma, M, H$ for $\Sigma = 0$ and then take the limit $(\Omega_M, \Omega_H) \to (0,0)$:

$$\lim_{(\Omega_M, \Omega_H) \to (0,0)} f_\Sigma(0, \Omega_M, \Omega_H) = -2\gamma L_2,$$

$$\lim_{(\Omega_M, \Omega_H) \to (0,0)} f_M(0, \Omega_M, \Omega_H) = -2\gamma L_2,$$

$$\lim_{(\Omega_M, \Omega_H) \to (0,0)} f_H(0, \Omega_M, \Omega_H) = 6\gamma L_2,$$

where we have introduced a notation

$$L_n = \lim_{(\Omega_M, \Omega_H) \to (0,0)} \frac{(\Omega_H)^n}{3\Omega_M + \Omega_H},$$

which will also be used later. When $\Omega_H > 0$, or equivalently $\lambda > 0$, we have $L_2 = 0$ and the fixed point F is always well defined as it should be. The existence of the limit is inconclusive when $\Omega_H < 0$ ($\lambda < 0$) and the orbit satisfies

$$\lim_{(\Omega_M, \Omega_H) \to (0,0)} \frac{\Omega_H}{\Omega_M} = -3. \tag{3.21}$$

However, such an orbit merely represents one that ends up in the singularity $2\lambda(H+\sigma)^2 = -1$. Since we have already discussed this case in detail, we exclude those orbits from our consideration here. As long as an orbit does not hit the singularity when approaching F, L_2 should exist and be equal to zero.

Since the right-hand sides can be evaluated only as a limit associated with each reference orbit, the linearisation takes an extra step. We first expand the equations around an arbitrary point $(\Sigma_0, \Omega_{M0}, \Omega_{H0})$ and then take the limit $(\Sigma_0, \Omega_{M0}, \Omega_{H0}) \to (0,0,0)$. The resultant linear equations are given as follows:

$$\frac{d\delta\Sigma}{d\alpha} = \frac{3}{2}(\gamma-2)\delta\Sigma + \left(2 + 6\gamma L_1^2\right)\delta\Omega_M + \frac{1}{2}\left[\gamma + 2 + 4\gamma L_1 (L_1 - 2)\right]\delta\Omega_H, \tag{3.22}$$

$$\frac{d\delta\Omega_M}{d\alpha} = \left(3\gamma - 4 + 6\gamma L_1^2\right)\delta\Omega_M + 2\gamma (L_1 - 1)^2 \delta\Omega_H, \tag{3.23}$$

$$\frac{d\delta\Omega_H}{d\alpha} = -18\gamma L_1^2 \delta\Omega_M - 2\left[2 + 3\gamma L_1 (L_1 - 2)\right]\delta\Omega_H, \tag{3.24}$$

where the δs preceding the variables denote their small perturbation. Unless the orbit is the singular one specified by (3.21), L_1 is finite and therefore these linearised equations are well defined. However, the asymptotic value of L_1 does depend on each orbit. Going back to its definition, one notices that

$$L_1 = \frac{\mathcal{R}}{\mathcal{R}+3},$$

where

$$\mathcal{R} = \frac{\Omega_H}{\Omega_M} = 6\lambda(H+\sigma)^2.$$

We already know its behaviour near F since we have

$$\mathcal{R} \sim 6\lambda H^2 \sim 2\lambda\rho.$$

Solving (3.6), we obtain

$$\mathcal{R} = \mathcal{R}_0 e^{-3\gamma\alpha}, \tag{3.25}$$

where $\mathcal{R}_0$ is an orbit-specific constant. We also note that

$$\delta\Omega_H = \mathcal{R}\delta\Omega_M \tag{3.26}$$

along each orbit, so that we have an additional linear constraint. Now, the linearized equations (3.22)-(3.24) can be written

$$\frac{d\delta\Sigma}{d\alpha} = \frac{3}{2}(\gamma-2)\delta\Sigma + \frac{1}{3+\mathcal{R}}\left(6 + \frac{1}{2}\mathcal{R}\left[10 + 3\gamma - \mathcal{R}(3\gamma-2)\right]\right)\delta\Omega_M, \tag{3.27}$$

$$\frac{d\delta\Omega_M}{d\alpha} = \delta\Omega_M\left[3\gamma - 4 + \frac{6\gamma\mathcal{R}}{3+\mathcal{R}}\right], \tag{3.28}$$

$$\frac{d\delta\Omega_H}{d\alpha} = \delta\Omega_H\left[-4 + \frac{6\gamma\mathcal{R}}{3+\mathcal{R}}\right]. \tag{3.29}$$

and we can easily read off the eigenvalues of the linearization matrix

$$\left(\frac{3}{2}(\gamma-2), 3\gamma - 4 + \frac{6\gamma\mathcal{R}}{\mathcal{R}+3}, -4 + \frac{6\gamma\mathcal{R}}{\mathcal{R}+3}\right). \tag{3.30}$$

Notice that the eigenvalues are orbit and time dependent through $\mathcal{R}(\alpha)$. As it was necessary to take a non-standard approach to obtain the eigenvalues, we will later confirm the validity of the result by performing numerical calculations.

3.3.2 Stability of the matter-dominated solution

The first of the eigenvalues (3.30) is negative and represents the stability of F against perturbation of Σ. Since the third eigenvalue is always smaller than the second, the condition for the stability is

$$3\gamma - 4 + \frac{6\gamma\mathcal{R}}{\mathcal{R}+3} < 0.$$

Let us first consider $\lambda > 0$, which corresponds to $\mathcal{R} > 0$. In this case, we have

$$0 < \frac{\mathcal{R}}{\mathcal{R}+3} < 1,$$

and consequently F is definitely stable for $\gamma < 4/9$, which means positive λ cannot be relevant in the context of inflation, and F is unstable for $\gamma > 4/3$. For $4/9 < \gamma < 4/3$, the stability is orbit-dependent. When an orbit satisfies

$$\mathcal{R} > \frac{12 - 9\gamma}{9\gamma - 4}, \tag{3.31}$$

it runs away from F. When $\mathcal{R}$ is smaller than this threshold value, the orbit is attracted towards F. Note that the value of $\mathcal{R}$ is time-dependent so that the stability can change over the course of the evolution. In particular, from (3.25), $\mathcal{R}$ is monotonically decreasing as long as the orbit stays close to the matter-dominated solution. We can immediately conclude that for orbits with $\mathcal{R} < (12 - 9\gamma)/(9\gamma - 4)$, the stability does not change as the universe expands. For those satisfying (3.31), they typically become stable asymptotically in the future since $\mathcal{R}$ can only increase when the universe is dominated by both Maxwell's and Horndeski's terms. For a physically interesting range of initial conditions, we shall later confirm that the instability of the electric field saturates before the orbit goes too far from F and eventually comes back to it.

For $\lambda < 0$, the dynamics is very different – depending on $\mathcal{R} \gtrless -3$. As was already mentioned, the critical value $\mathcal{R} = -3$ corresponds to the singularity $2\lambda(H + \sigma)^2 = -1$ discussed in section 3.1.1. Firstly, $\mathcal{R} \in (-3, 0)$ implies $\mathcal{R}/(\mathcal{R} + 3) \in (-\infty, 0)$ and therefore the orbits in this range are stable as long as $\gamma < 4/3$. For $\mathcal{R} < -3$, we have $\mathcal{R}/(\mathcal{R} + 3) \in (1, \infty)$ and $\mathcal{R}$ monotonically increases in the vicinity of F. It approaches $\mathcal{R} = -3$ from below and therefore any orbit eventually becomes unstable. Since we already know $\mathcal{R} = -3$ is the singularity, we conclude that the orbits with this range of initial $\mathcal{R}$ can never settle down at F, regardless of the equation of state parameter γ; see figure 3 for simulation of an inflationary universe with initial conditions satisfying $\mathcal{R} = -100$. The solution approaches F until $\mathcal{R}$ is close to the critical value in which case the universe moves away from F. After leaving F, we expect the system enters the regime where Ω_m is dynamically negligible. Then the analysis in the section 3.1.1 applies and the singularity is inevitable.

3.3.3 Dynamics with $\lambda > 0$ in a radiation- or dust-dominated universe

We have found that the theory is quite innocuous for positive non-minimal coupling constant, λ, in which case there is no singularity and the Friedmann solution F is stable at late times. However, as shown above, there is a transient period where F is unstable for fluids satisfying $\gamma > 4/9$. In this period, Ω_M and Ω_H grow and, if the instability does not saturate before they become too large, the universe will eventually become strongly anisotropic. Since this introduces potential problems in the radiation or dust-dominated epoch, it is of interest to specify the range of initial conditions such that the universe is close to F at all the subsequent times, i.e., $|\Omega_m - 1| \ll 1$. In this subsection, therefore, we investigate the dynamics close to F in more detail for dust ($\gamma = 1$) and radiation ($\gamma = 4/3$) with a positive non-minimal coupling constant ($\lambda > 0 \Leftrightarrow \Omega_H > 0$).
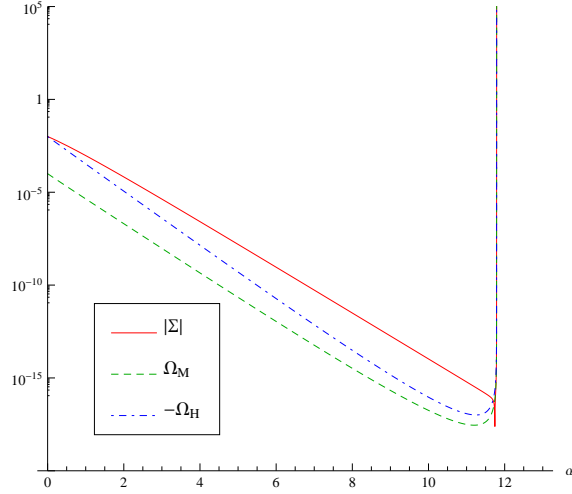
Figure 3. Simulation of (3.13)-(3.15) for an inflationary universe with $\gamma = 0.1$ and initial conditions close to F ($\Sigma = -0.01$, $\Omega_M = 0.0001$, $\Omega_H = -0.01$). The singularity at $3\Omega_M + \Omega_H = 0$ occurs at time $\alpha = 11.65$.

We can easily solve the linearized equations (3.28) and (3.29) exactly:

$$\delta\Omega_M(\alpha) = \delta\Omega_{M0} \left(\frac{3+\mathcal{R}_0}{3+\mathcal{R}}\right)^2 e^{-(4-3\gamma)\alpha}, \tag{3.32}$$

$$\delta\Omega_H(\alpha) = \delta\Omega_{H0} \left(\frac{3+\mathcal{R}_0}{3+\mathcal{R}}\right)^2 e^{-4\alpha}, \tag{3.33}$$

where $\delta\Omega_{M0}$ and $\delta\Omega_{H0}$ are integration constants. At late times, when $\mathcal{R} \propto e^{-3\gamma\alpha} \to 0$, the Maxwell and Horndeski densities decay as $\delta\Omega_M \propto e^{-(4-3\gamma)\alpha}$ and $\delta\Omega_H \propto e^{-4\alpha}$. This is consistent with F being an attractor at late times as shown above. When $\mathcal{R} \gg 3$, the Maxwell and Horndeski densities grow as $\delta\Omega_M \propto e^{(9\gamma-4)\alpha}$ and $\delta\Omega_H \propto e^{2(3\gamma-2)\alpha}$. We note that the maximum values of $\delta\Omega_M$ and $\delta\Omega_H$ occur when $\mathcal{R} = (12 - 9\gamma)/(-4 + 9\gamma)$ and $\mathcal{R} = 6/(3\gamma - 2)$, respectively. Taking into account the linear constraint (3.26), it follows that the maximum values of both Ω_M and Ω_H are roughly equal. Now we find that the initial conditions must satisfy

$$\delta\Omega_{H0} \ll (\delta\Omega_{M0})^{\frac{1}{2}} \quad \Leftrightarrow \quad \lambda \ll (E_0 H_0)^{-1} \tag{3.34}$$

in a radiation-dominated universe and

$$\delta\Omega_{H0} \ll (\delta\Omega_{M0})^{\frac{2}{5}} \quad \Leftrightarrow \quad \lambda \ll (E_0)^{-\frac{6}{5}} (H_0)^{-\frac{4}{5}} \tag{3.35}$$

in a dust dominated universe to ensure that $|\Omega_m - 1| \ll 1$ at all the subsequent times. In figures 4 and 5, we show a numerical integration of the full non-linear equations (3.13)-(3.15) for radiation and dust, respectively. Since the initial conditions just barely satisfy the conditions (3.34)-(3.35), the peak values of Ω_H and Ω_M are at the one-percent level

of the total energy budget. Note that, to good accuracy, we have $\Sigma \lessgtr 0$ when $\Omega_M \lessgtr \Omega_H$. This introduces the possibility of cancelling the effects of spatial anisotropy on the Cosmic Microwave Background (CMB), which will be discussed in section 5. Under the assumption that the theory describes a generalised electrodynamics, we show in section 5 that the amplification of the electric field must come to an end before the start of big bang nucleosynthesis, when the temperature is $T \simeq 1$ MeV. This still leaves the possibility open for a huge amplification of electric fields in the period between inflationary reheating and the nucleosynthesis. In the period of amplification, the electric field grows very quickly, $\Omega_M \propto e^{8\alpha}$. After the peak value is reached, Ω_H rapidly decays and soon becomes negligible. At that stage, the dynamics becomes similar to the conventional electrodynamics; Ω_M decays logarithmically (constant at the linear level) until the dust-dominated epoch when it decays as $\Omega_M \propto e^{-\alpha}$.
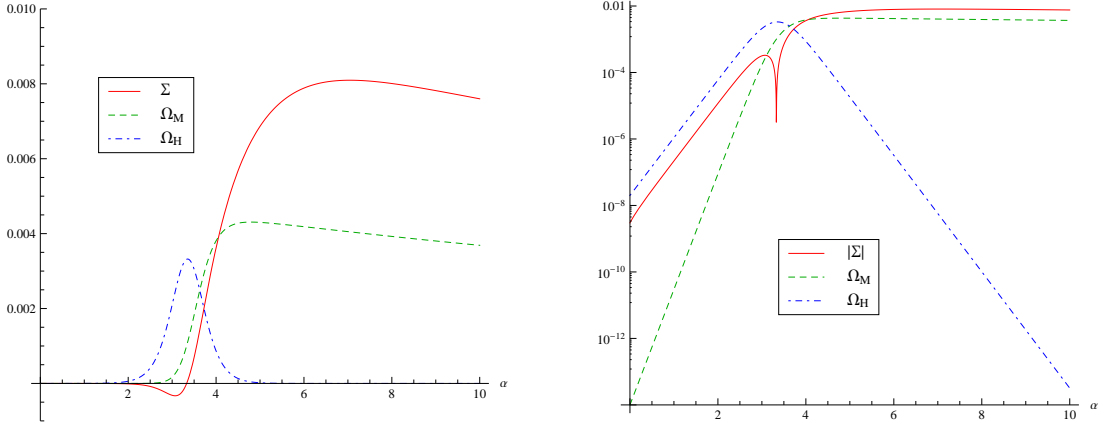

Figure 4. Simulation of (3.13)-(3.15) for radiation ($\gamma = 4/3$) and a positive non-minimal coupling coupling constant ($\lambda > 0$). Initial conditions ($\Sigma = -3*10^{-9}$, $\Omega_M = 10^{-14}$, $\Omega_H = 2*10^{-8}$) are such that the orbit is always close to the Friedmann solution F. From the logarithmic plot to the right it is clear that the ratio Ω_H/Ω_M is monotonically decaying in agreement with equation (3.25).

3.4 Stability of the other fixed points

Given the complexity of the dynamics, it is also helpful to analyse the stability of the other fixed points. Here we present the eigenvalues for K_+, M and H_E. The linear stability analysis for K_- is inconclusive since we have $H + \sigma = 0, E = 0$ there, and so the Horndeski energy density $3\lambda(H + \sigma)^2 E^2$ is generically second order in perturbation. Our numerical simulations in the next subsection indicate that K_- is a past attractor for $\lambda > 0$.

When either Ω_M or Ω_H is nonzero, the stability analysis is straightforward since there is no orbit-dependence. We obtain the following eigenvalues:

Fixed point M

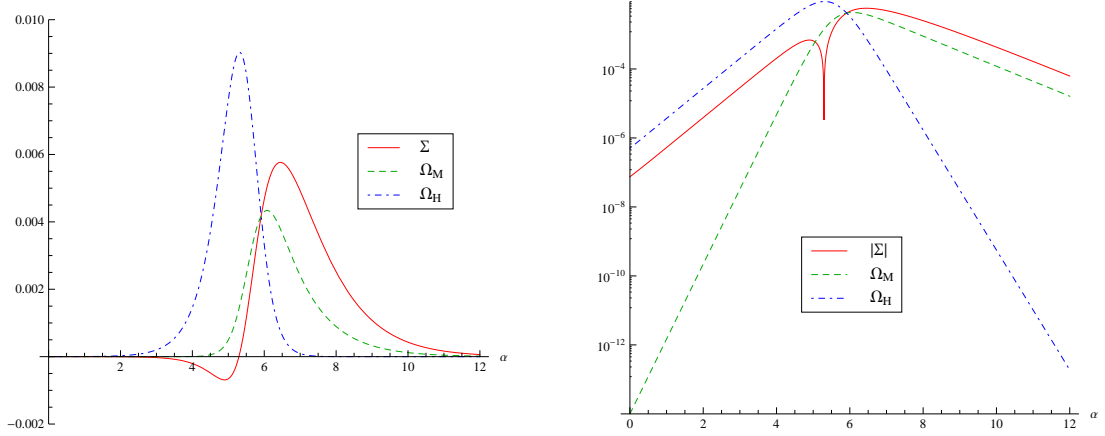$$\left(-\frac{3}{4} \left[2 - \gamma \pm \sqrt{(2 - \gamma)(3\gamma^2 - 17\gamma + 18)} \right], -3\gamma \right),$$

– 15 –

Figure 5. Simulation of (3.13)-(3.15) for dust ($\gamma = 1$) and a positive non-minimal coupling coupling constant ($\lambda > 0$). Initial conditions ($\Sigma = -7.5 * 10^{-8}$, $\Omega_M = 10^{-14}$, $\Omega_H = 5 * 10^{-7}$) are such that the orbit is always close to the Friedmann solution F. From the logarithmic plot to the right it is clear that the ratio Ω_H/Ω_M is monotonically decaying in agreement with equation (3.25).

Fixed point H_E

$$(\pm 2, 2 - 3\gamma).$$

Whenever M exists ($\gamma > 4/3$), it is a future attractor. The new fixed point H_E is a saddle regardless of γ, and the orbit is temporarily attracted towards it for initial conditions close to F that fail to satisfy the conditions (3.34)-(3.35). We did not find the dynamics around this solution to be of any phenomenological interest.

The eigenvalues of K_+ is dependent on each reference orbit and the linearisation can be carried out in a similar way as for F in section 3.1.1. In the end, we derive the following equations:

$$\frac{d\delta\Sigma}{d\alpha} = 3(2 - \gamma)\delta\Sigma + \frac{24 - 9\gamma + \mathcal{R}\left[2\mathcal{R} + 38 - 3\gamma(\mathcal{R} + 4)\right]}{2(\mathcal{R} + 3)}\delta\Omega_M,$$
$$\frac{d\delta\Omega_M}{d\alpha} = \frac{10\mathcal{R} - 6}{\mathcal{R} + 3}\delta\Omega_M,$$
$$\frac{d\delta\Omega_H}{d\alpha} = \frac{4\mathcal{R} - 24}{\mathcal{R} + 3}\delta\Omega_H,$$

and read off the eigenvalues:

$$\left(3(2 - \gamma), \frac{10\mathcal{R} - 6}{\mathcal{R} + 3}, \frac{4\mathcal{R} - 24}{\mathcal{R} + 3}\right).$$

The first eigenvalue is associated with the perturbation of Ω_m and positive so that K_+ cannot be a future attractor. The time-dependence of $\mathcal{R} = 6\lambda(H + \sigma)^2$ around K_+ is given by evaluating (3.9) with $H \sim \sigma$ as

$$\mathcal{R} \sim \mathcal{R}_0 e^{-6\alpha}.$$

For $\mathcal{R} < 0$, the situation is analogous to F except that K_+ is irrelevant to future asymptotic behaviour. An orbit with $\mathcal{R} < -3$ initially reaches the singularity while K_+ is a saddle point for the others. For $\mathcal{R} > 0$, the stability changes over the course of the evolution.

3.5 Numerical Analysis

We conclude this section by showing two visualizations that confirm the analysis of this section, and suggest that K_- is a past attractor (which is the case in the minimally coupled theory) for $\lambda > 0$. Figure 6 shows a phase portrait of the invariant subset $\Omega_m = 0$ (a universe without the fluid). This subset is effectively two dimensional by the Friedmann equation (3.11) and the phase space is completely characterised by the variables Σ and Ω_M. The green shaded region corresponds to $\lambda > 0$. It clearly shows that K_- is a past attractor for $\lambda > 0$ on this subspace. Note the existence of stream lines directed towards K_- in the $\lambda < 0$ region indicating orbit dependence of the stability for a negative non-minimal coupling constant. In the region $\lambda < 0$, there are two distinct flows inside and outside the bold red lines that denote the location of the singularity. The outer region is marked with a red mesh and corresponds to the pathological region (3.17) where the singularity is inevitable. Note that (3.17) can be written $1 + 2\Omega_M < \Sigma^2 + \Omega_m$ which explains why the entire region $\Sigma^2 < 1$ is non-pathological on the subspace $\Omega_m = 0$. When a fluid is included, however, the singularity can be reached even from a small initial shear (for example figure 3). Figure 7 shows the full three-dimensional phase flow for a few orbits with $\lambda > 0$, which demonstrates the past-stability of K_- and confirms the time-dependent stability of F.

4 Dynamics of a magnetic field in axisymmetric Bianchi type I

In the usual electromagnetism, Maxwell's equations treat electric and magnetic fields in a symmetric manner. In the context of Bianchi cosmologies, source-free pure electric and magnetic fields are mathematically indistinguishable. The modification $\mathcal{L}_H$ breaks this duality. Let us consider a homogeneous magnetic field along x-axis in the spacetime (3.1). In parallel to the electric field of the previous section, we define B as the magnetic field seen by a comoving observer:

$$\mathbf{F} \equiv \frac{1}{2} F_{ab} \, dx^a \wedge dx^b = B(t)(e^{\alpha+\beta} dy) \wedge (e^{\alpha+\beta} dz). \qquad (4.1)$$

Contrary to the electric case, equation (2.7) is trivially satisfied for this Faraday tensor. Instead the evolution of the comoving magnetic field is given by the Bianchi identity $d\mathbf{F} = 0$ which leads to

$$\dot{B} = -2 (H + \sigma) B. \qquad (4.2)$$

In contrast to the peculiar dynamics of the electric field, there is no modification to the evolution equation for the magnetic field since it comes from the closedness of the field-strength 2-form. Then, we can solve it easily to obtain
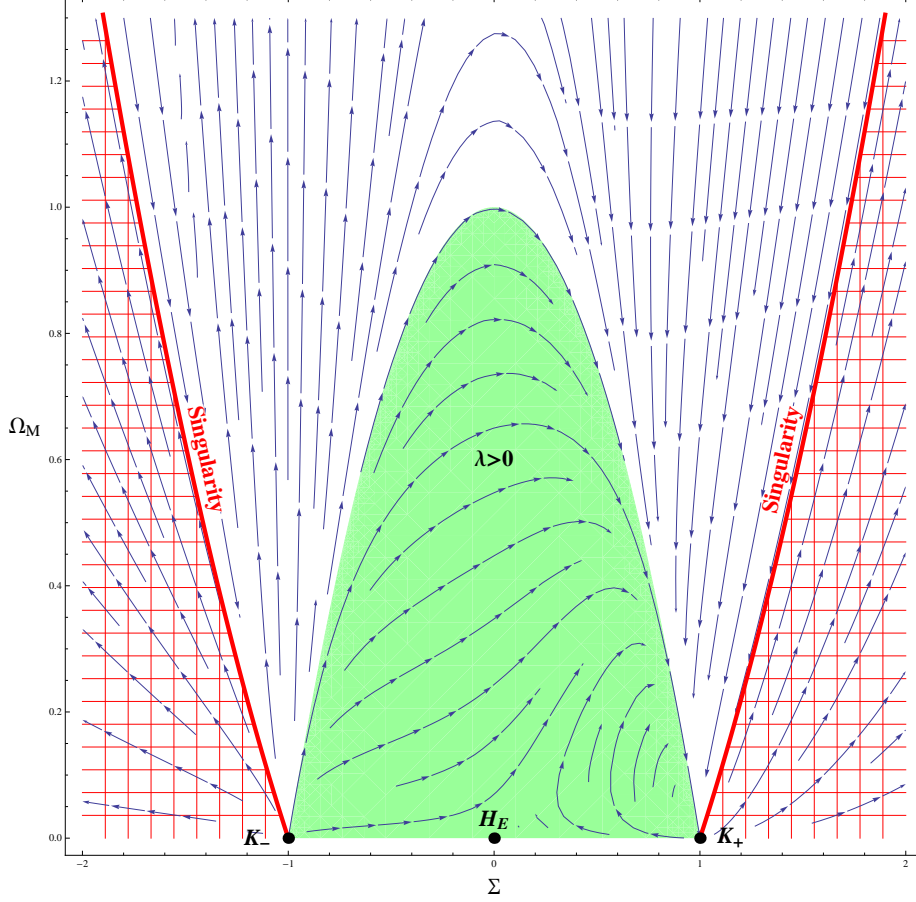
$$B = \frac{B_0}{e^{2(\alpha+\beta)}}. \qquad (4.3)$$

Figure 6. Phase flow for the subsystem $\Omega_m = 0$ in the electric case. The green shaded region corresponds to $\lambda > 0$. The red bold line is the position of the singularity, while the red mesh is the region where the singularity is inevitable.

This simply means the magnetic field is adiabatically decaying due to the expansion of the universe.

The Einstein equations can be written in the following convenient from:

$$H^2 = \sigma^2 + \frac{1}{3}\rho + \frac{1}{6}B^2 - \frac{2}{3}\lambda\left(H + \sigma\right)\left(H - 2\sigma\right)B^2, \tag{4.4}$$

$$\dot{H} + H^2 = -\frac{1}{1 + \lambda B^2}\left[H^2 + H\sigma + 2\sigma^2 + \frac{2}{3}\left(\gamma - 1\right)\rho - 2\lambda H\left(H + 2\sigma\right)B^2\right] \tag{4.5}$$

$$+ \frac{1}{2(1 + \lambda B^2)^2}\left[\left(H + \sigma\right)^2 + \frac{\gamma - 1}{3}\rho - \frac{1}{6}B^2 - \frac{4}{3}\lambda\left(H + \sigma\right)^2 B^2\right],$$

$$\dot{H} + \dot{\sigma} = -\frac{1}{1 + \lambda B^2}\left[3\sigma\left(H + \sigma\right) - 3\lambda H\left(H + \sigma\right)B^2 + \frac{\gamma}{2}\rho\right]. \tag{4.6}$$

Again, there appears to be a problem for negative λ when $\lambda B^2 \sim -1$. This time, the singularity stems from the Einstein equations instead of the evolution equation for F_{ab} as in the electric case. From (4.6), it is clear that $\dot{H} + \dot{\sigma}$ is positive definite in an expanding
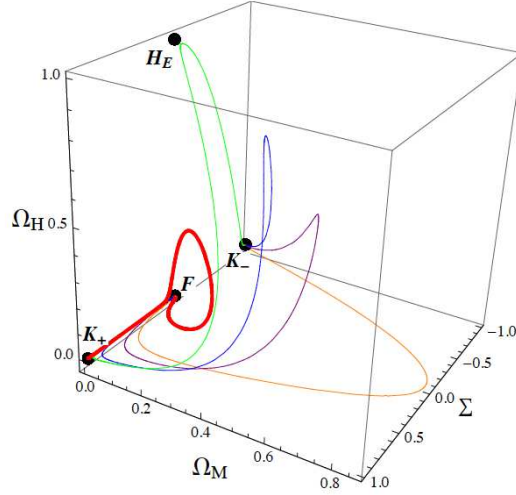
Figure 7. Simulations of the non-linear equations (3.13)-(3.15) for a radiation fluid ($\gamma = 4/3$). All the orbits correspond to a positive non-minimal coupling, $\lambda > 0$. The thick red line starts from K_+, gets repelled from F and eventually comes back to it, which confirms the time-dependence of the stability for F. All the other orbits traverse from K_- to F indicating K_- is a past attractor.

universe $H > 0$ for initial conditions satisfying

$$H + \sigma > 0, \quad -\lambda B^2 > 1. \tag{4.7}$$

According to (4.3), B^2 decays in this regime and consequently the system has to reach the singularity $\lambda B^2 = -1$. As one can see in figure 8, the shear becomes negative before reaching the singularity. The behaviour is insensitive to the initial conditions or value of λ whenever $-\lambda B^2 > 1$ initially. This condition coincides with the one for the Horndeski modified term to have a significant contribution to the dynamics. We conclude that $\lambda < 0$ is pathological for the magnetic case too.

While there is no divergent behaviour for positive λ, the energy density of the magnetic field is not necessarily positive, in contrast to the electric field case. This will not cause any problem for ordinary expanding universes with $H + \sigma > 0$ since the solution (4.3) ensures monotonic decrease of B. However, it may result in unusual behaviours when we go backwards in time. We repeat the dynamical system analysis of the previous section and show that $\lambda > 0$ is cosmologically viable.

4.1 Expansion-normalised autonomous system

We introduce the following normalised variables:

$$\Sigma = \frac{\sigma}{H}, \qquad \Omega_M = \frac{B^2}{6H^2}, \qquad \Omega_H = -\frac{2}{3}\lambda\left(1 + \Sigma\right)\left(1 - 2\Sigma\right)B^2, \qquad \Omega_m = \frac{\rho}{3H^2}.$$

The normalised Friedmann equation takes the canonical form

$$1 = \Sigma^2 + \Omega_H + \Omega_M + \Omega_m. \tag{4.8}$$
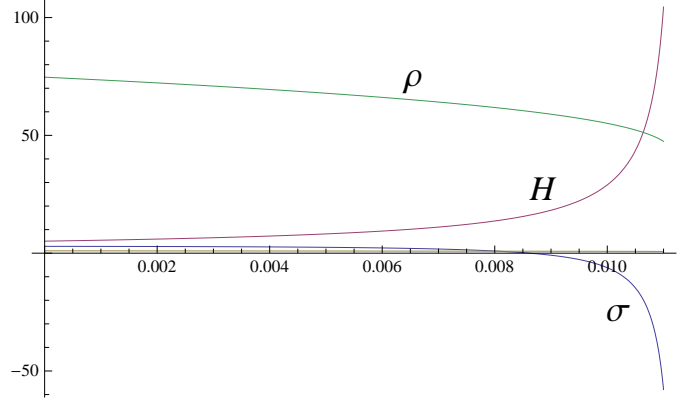
$-19-$

Figure 8. Occurrence of the singularity for $\lambda = -2, \gamma = 1$. Initial conditions are $H = 5.1, \sigma = 2.9, B = 1$. In fact, the dynamics is more or less the same for any values of parameters or initial conditions as long as $-\lambda B^2 > 1$. In this case, while the magnetic field is subdominant in the Friedmann equation (4.4), it still affects the evolution of H and σ and causes their divergences. Note that $\sigma/H \to -1$ as the singularity is approached. This implies the approach towards the singularity appears as a flow into K_- in the expansion-normalised variables.

In contrast to the electric field, Ω_H is not positive definite regardless of the sign of λ. Following the steps of the previous section, we rewrite (4.5) as a defining equation for the deceleration parameter:

$$q = -\frac{2(1+\Sigma)(1-2\Sigma)}{3\Omega_H - 2(1+\Sigma)(1-2\Sigma)}\left[1 + \Sigma + 2\Sigma^2 + 2(\gamma-1)\Omega_m + \frac{3(1+2\Sigma)}{(1+\Sigma)(1-2\Sigma)}\Omega_H\right]$$
$$-\frac{2(1+\Sigma)^2(1-2\Sigma)^2}{[3\Omega_H - 2(1+\Sigma)(1-2\Sigma)]^2}\left[(1+\Sigma)^2 + (\gamma-1)\Omega_m - \Omega_M + \frac{2(1+\Sigma)}{1-2\Sigma}\Omega_H\right].$$

We derive the following evolution equations for the normalised variables:

$$\frac{d\Sigma}{d\alpha} = (1+q)(1+\Sigma) \tag{4.9}$$
$$+\frac{6(1+\Sigma)(1-2\Sigma)}{3\Omega_H - 2(1+\Sigma)(1-2\Sigma)}\left[\Sigma(1+\Sigma) + \frac{\gamma}{2}\Omega_m + \frac{3}{2(1-2\Sigma)}\Omega_H\right],$$

$$\frac{d\Omega_M}{d\alpha} = 2(q - 1 - 2\Sigma)\Omega_M, \tag{4.10}$$

$$\frac{d\Omega_H}{d\alpha} = -4(1+\Sigma)\Omega_H - \frac{1+4\Sigma}{(1+\Sigma)(1-2\Sigma)}\Omega_H\frac{d\Sigma}{d\alpha}, \tag{4.11}$$

$$\frac{d\Omega_m}{d\alpha} = (2q - 3\gamma + 2)\Omega_m. \tag{4.12}$$

4.2 Fixed points and their stabilities

We repeat the standard stability analysis. The structure of the state space is analogous to the electric case. There are four fixed points residing in the conventional magnetic Bianchi type I, and another with non-vanishing Ω_H.

Flat Friedmann universe : F

$$(\Sigma, \Omega_M, \Omega_H, \Omega_m) = (0, 0, 0, 1)$$

with eigenvalues

$$\left(-4, -\frac{3}{2}(2 - \gamma), 3\gamma - 4\right). \tag{4.13}$$

Since the evolution equation of the magnetic field is well-behaved, the linearisation can be carried out as usual. The stability does not change from the usual magnetic cosmologies. F is a future attractor for $\gamma < 4/3$. Note the zero eigenvalue in the radiation case $(\gamma = 4/3)$.

Magnetic Bianchi type I : B

$$(\Sigma, \Omega_M, \Omega_H, \Omega_m) = \left(\frac{3\gamma - 4}{4}, \frac{3}{16}(2 - \gamma)(3\gamma - 4), 0, \frac{3}{8}(4 - \gamma)\right)$$

with eigenvalues

$$\left(-\frac{3}{4}\left(2 - \gamma \pm \sqrt{(2 - \gamma)(3\gamma^2 - 17\gamma + 18)}\right), -3\gamma\right). \tag{4.14}$$

The existence condition is $\gamma > 4/3$ and it is always stable.

Kasner solutions : $K_\pm$

$$(\Sigma, \Omega_M, \Omega_H, \Omega_m) = (\pm 1, 0, 0, 0).$$

The eigenvalues for K_+ are easily computed as

$$(3(2 - \gamma), -2, -8), \tag{4.15}$$

indicating it is a saddle while being a future attractor for the subsystem $\Omega_m = 0$. For K_-, we have the same indeterminacy of the linear stability as was encountered in the electric case. It will be examined by numerical analysis later.

Ω_H-dominated universe : H_B

$$(\Sigma, \Omega_M, \Omega_H, \Omega_m) = \left(\frac{1}{2}, 0, \frac{3}{4}, 0\right).$$

This fixed point does not itself represent a physical solution since it corresponds to the limit $-\lambda(H + \sigma)(H - 2\sigma) \to \infty$. As in the electric case, this does not mean that it is dynamically irrelevant since it can be reached asymptotically from finite H and σ (see figure 11). Note that $q = 0$ for this fixed point which means that the back-reaction of the vector field exactly cancels out like for the Horndeski dominated solution in the electric case. Its eigenvalues are given by

$$(-12, -3(\gamma + 2), 6). \tag{4.16}$$

Therefore it is always a saddle point.

4.3 Numerical analysis

Since the uniform magnetic field decays adiabatically, the evolution close to the Friedmann solution F is trivial compared to the electric case. Near F, Ω_M evolves as in the conventional electromagnetism; logarithmically decaying (constant at the linear level) in the radiation era ($\gamma = 4/3$) because of the zero eigenvalue, and in proportion to $e^{-\alpha}$ in the dust era ($\gamma = 1$), see [79–81]. It follows immediately from its definition that $\Omega_H \propto B^2 \propto e^{-4\alpha}$, i.e., the evolution is independent of the equation of state of the perfect fluid. See figure 9 for a simulation of the full non-linear equations (4.9)-(4.11) close to F.

It is again helpful to visualise the global structure of the phase space. Figure 10 shows a phase portrait of the invariant subset $\Omega_m = 0$. The green shaded region corresponds to $\lambda > 0$. It is clearly seen that K_- is a past attractor for $\lambda > 0$ on this subspace. Note the existence of stream lines directed towards K_- in the region $\lambda < 0$. They indicate orbit dependence of the stability for a negative non-minimal coupling constant value. The red mesh corresponds to the pathological region (4.7) where a singularity, marked with the red bold line, is inevitable. Note, from its definition, that $\Omega_H = 0$ when $\Sigma = 1/2$. Therefore, although it has no physical significance, the point $(\Omega_M, \Sigma) = (3/4, 1/2)$ appears as if it were a fixed point. Figure 11 shows the full three-dimensional phase flow for a few orbits with $\lambda > 0$. Overall, our simulations indicate that K_- is a past attractor for $\lambda > 0$.
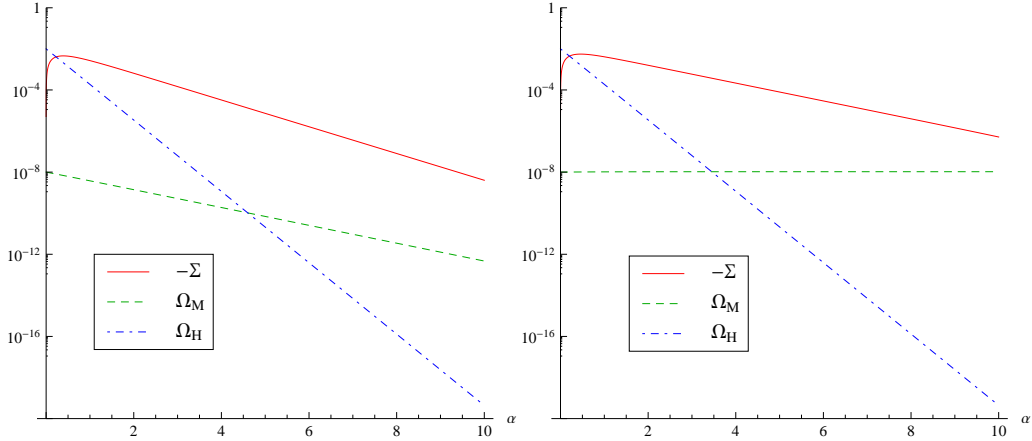

Figure 9. Simulation of (4.9)-(4.11) close to the Friedmann solution F for dust ($\gamma = 1$) to the left and radiation ($\gamma = 4/3$) to the right. Initial conditions: $\Sigma = 10^{-8}$, $\Omega_M = 10^{-8}$, $\Omega_H = 10^{-2}$.

5 Observational constraints

In this section, we shall discuss the possibility of detecting any signature of the theory with a positive coupling constant λ in the standard cosmology. Since the correction term comes with a factor of the space-time curvature which is tiny in the units of Planck mass,
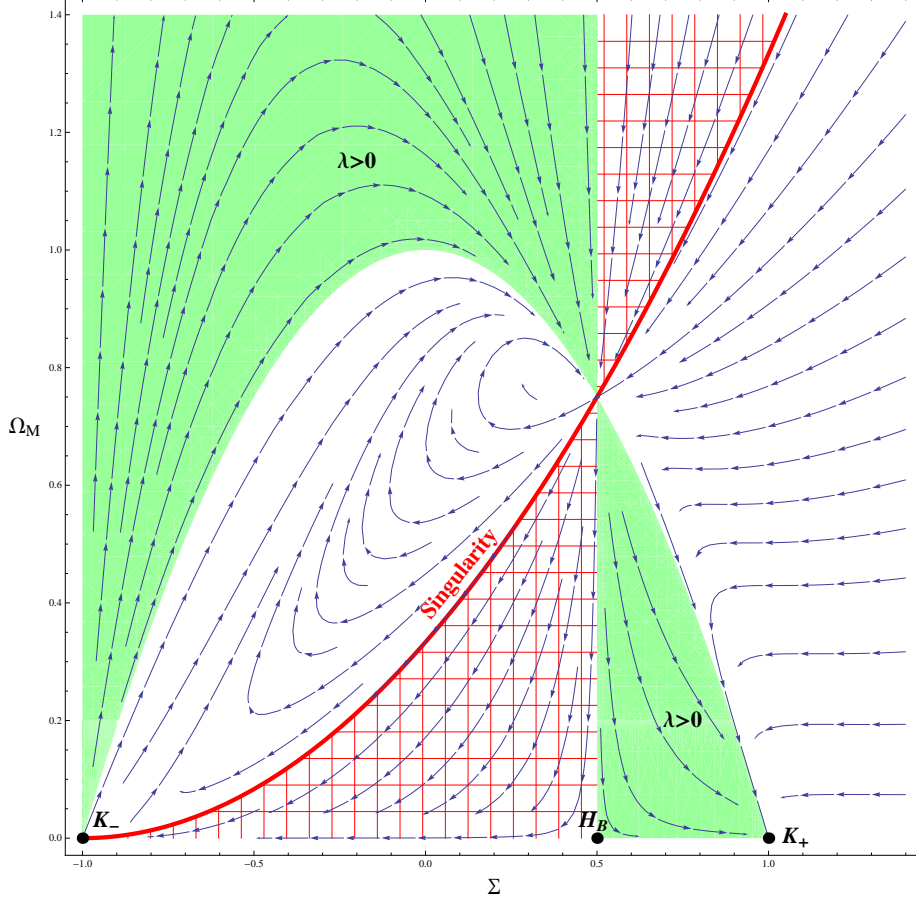
Figure 10. Phase flow for the subsystem $\Omega_m = 0$ in the magnetic case. The green shaded region corresponds to $\lambda > 0$. The red bold line is the position of the singularity, while the red mesh is the region where the singularity is inevitable. In the meshed region, the flow goes towards K_-. For $\lambda > 0$, K_- appears to be a past attractor.

one expects that a large value of the parameter λ is required to see any appreciable effect. Combining the constraint on spatial anisotropy coming from big bang nucleosynthesis [82] and the validity of the Maxwellian electromagnetism in terrestrial environments, we will show that the growth of the electric effect discussed in section 3.3.3 must have come to an end before the CMB temperature dropped below $T \sim 0.1$ MeV. After that time, Ω_H decays much faster than Ω_M, and so the effect of the Horndeski Lagrangian term is completely negligible at late times. Since we restrict ourselves to the dynamics around FLRW in this context, the magnetic case will not be discussed in detail due to its trivial evolution seen in the previous section.

A non-minimal coupling of electrodynamics to gravity is known to cause a number of observational effects, such as frequency-dependent bending of light in gravitational fields
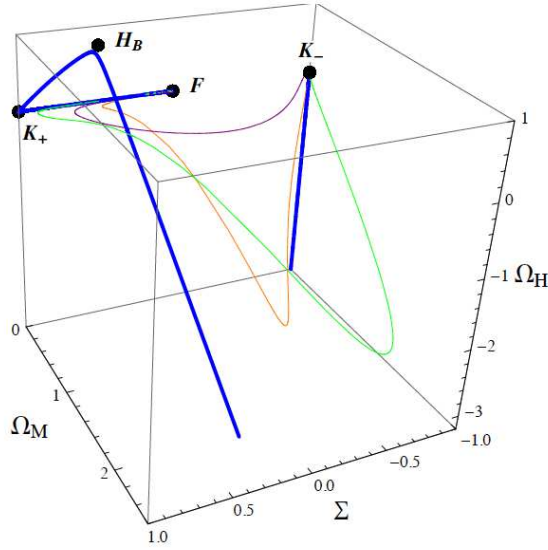
Figure 11. Simulations of the non-linear equations (4.9)-(4.11) for a radiation fluid ($\gamma = 4/3$). All orbits correspond to a positive non-minimal coupling, $\lambda > 0$. For each orbit, phase flow is in the direction towards the Friedmann solution F, and initial conditions are close to K_- with non-vanishing perfect fluid and the vector field, which underwrites the past stability of K_-. The thick blue orbit temporarily leaves the boundary of the box. After it returns, it has transient periods close to H_B and K_+ before settling down at F.

[83]. Since those effects must be suppressed in terrestrial environments, we require

$$|\mathcal{L}_H| \ll \left| -\frac{1}{4}F^2 \right| \tag{5.1}$$

evaluated around the Earth. In an orthonormal basis, the components of the Riemann tensor for the Schwarzschild solution are of order $R_{abcd} \sim R_s/r^3$, where R_s is the Schwarzschild radius and r is the radial coordinate in Schwarzschild coordinates. Inserting the radius and mass of the Earth, we find that the constraint (5.1) is equivalent to

$$|\lambda| \ll \frac{r^3}{R_s} M_{pl}^2 \sim 10^{90}, \tag{5.2}$$

in which the upper limit appears enormous since we have used a large mass scale (M_{pl}) to normalise the non-minimal coupling constant (which is a reasonable convention in the context of modified gravity). From the linear analysis in section 3.3.3, we know that both Ω_M and Ω_H start to decay when

$$\mathcal{R} \equiv \frac{\Omega_H}{\Omega_M} \simeq \lambda \frac{H^2}{M_{pl}^2} \tag{5.3}$$

is close to unity. At the time $\mathcal{R} \sim 1$, the expansion rate must satisfy

$$H \gg 10^{-45} M_{pl} \tag{5.4}$$

to be consistent with the constraint (5.2). Using the approximation $T \sim (H M_{pl})^{\frac{1}{2}}$, where T is the CMB temperature, this imposes the following condition on the temperature at $\mathcal{R} \sim 1$:

$$T_{\mathcal{R} \sim 1} \gg 10^{-23} M_{pl} \sim 0.1 \text{ MeV}. \tag{5.5}$$

Thus, to be consistent with the local constraint on terrestrial electromagnetism, the growth of the normalised vector field energy is required to end at latest around the time of big bang nucleosynthesis when $T_{\text{bbn}} = 0.1$-1 MeV. Since $\mathcal{R}$ is monotonically decreasing in the perfect fluid dominated universe ($\mathcal{R} \propto e^{-4\alpha}$ in the radiation-dominated epoch and $\mathcal{R} \propto e^{-3\alpha}$ in the dust-dominated epoch), we set $\mathcal{R} \ll 1$ for $T \ll 1$ MeV. It follows that the energy density of the vector, both in the electric and the magnetic case, will evolve as in the conventional electrodynamics, i.e. $E^2, B^2 \propto e^{-4\alpha}$ and therefore $\Omega_H \propto e^{-4\alpha}$. Using that $T \propto e^{-\alpha}$, we obtain the relation

$$\Omega_H^{\text{bbn}} \simeq \Omega_H^0 \left(\frac{T_{\text{bbn}}}{T_0} \right)^4, \tag{5.6}$$

where Ω_H^0 and Ω_H^{bbn} are the values of Ω_H today and at big bang nucleosynthesis, respectively. The CMB temperature today is $T_0 \simeq 10^{-4}$eV. We therefore have $\Omega_H^{\text{bbn}} \simeq 10^{40} \Omega_H^0$. The theory of big bang nucleosynthesis requires that the universe was dominated by the CMB and three effectively massless neutrino species at T_{bbn}, i.e. that Ω_H^{bbn} must be smaller than unity. This corresponds to the following constraint at the present time:

$$|\Omega_H^0| < 10^{-40}. \tag{5.7}$$

We conclude that the effect of Ω_H on the evolution of the universe becomes increasingly less significant after the big bang nucleosynthesis. It is interesting to compare this constraint to that on a uniform magnetic field in the conventional electrodynamics. In that case the strongest limit on a uniform magnetic field today comes from the CMB anisotropy and is [84]

$$B_0 \lesssim 10^{-9} \text{ Gauss} \tag{5.8}$$

or equivalently

$$\Omega_M^0 = \frac{B_0^2}{6 M_{pl} H_0^2} \lesssim 10^{-12}, \tag{5.9}$$

which is much weaker than the upper bound on the Horndeski part of the Lagrangian (5.7). Evaluating these constraints at decoupling time ($z \simeq 1100$)

$$|\Omega_H^{\text{dc}}| \simeq \Omega_H^0 1100^4 \lesssim 10^{-28}, \qquad \Omega_M^{\text{dc}} \simeq \Omega_M^0 1100 \lesssim 10^{-9}, \tag{5.10}$$

we note that $|\Omega_H^{\text{dc}}| \ll \Omega_M^{\text{dc}}$. Consequently the additional anisotropic stress induced by the non-minimal coupling is completely negligible in the entire period between decoupling time and today. Now, since the upper bound on a uniform magnetic field comes from the modified temperature pattern in the CMB created by the shear (which is sourced by the anisotropic stress), it is clear that the constraint so far obtained is strong enough to exclude any signature of Ω_H in the CMB, and (5.9) holds also in Horndeski's generalised theory. This result reflects the fact that Ω_H decays much faster than Ω_M after reaching $\mathcal{R} \sim 1$.

In a similar manner, even a direct detection of homogenous cosmological magnetic fields would not improve the constraint on λ either since the regime $\Omega_H > \Omega_M$ must come prior to big bang nucleosynthesis.

So far we have assumed that the vector field is identified with photons, i.e. that the model is an extension of the ordinary electrodynamics. If this is not the case, and the model is merely taken to be a hypothetical vector-tensor theory, the bound on the non-minimal coupling constant (5.2) is not valid and the normalised vector field energy can grow also at late times if λ is enormous. To good approximation, we have $\Sigma \gtrless 0$ when $\Omega_M \gtrless \Omega_H$, as noted in section 3.3.3, and this could have some interesting phenomenological consequences. Since the $\Delta T/T$ created in the CMB by the shear is an integrated effect, which is proportional to [85]

$$\int_{\alpha_{\mathrm{dc}}}^{\alpha_0} \Sigma d\alpha,$$

a cancellation effect will occur if the peak value of the normalised vector field energy (around $\mathcal{R} \sim 1$) occurs between decoupling time (α_{dc}) and today (α_0). First, one will have a period with $\Omega_H > \Omega_M$ where both Ω_M and Ω_H grow and $\Sigma < 0$, followed by a period with $\Omega_H < \Omega_M$ where both Ω_M and Ω_H decay and $\Sigma > 0$; see figure 5. By fine-tuning the non-minimal coupling constant, the cancellation can be made exact leaving no effect on the CMB despite having a relatively large amplitude of shear. Numerical simulations show that this cancellation requires that Ω_H peaks at a low redshift; as an example, Ω_H must peak at redshift $z \sim 0.6$ for the initial conditions in figure 5. The shear still needs to be consistent with the supernovae Ia data which is isotropic to the 1% level, but this is significantly relaxed compared to the CMB bounds in models where Σ is positive or negative definite where $|\Sigma|$ is never greater than $\sim 10^{-5}$. The possibility of such a cancellation effect was mentioned in [86, 87] and to the best of our knowledge, Horndeski's theory is the first instance of a Lagrangian for which such a scenario may be dynamically realised.

6 Conclusion and outlook

In this work, we have explored the cosmological consequences of a most general vector-tensor theory that gives second-order field equations equivalent to Maxwell's equations when evaluated in flat space-time. We have focused on a simple dynamical setup which is well understood in the conventional electromagnetism, namely the case of a uniform electric or magnetic field in an axisymmetric Bianchi type I universe. We have investigated the non-linear evolution equations by using several different approaches, including the conventional dynamical systems analysis in terms of Hubble normalised variables such as Ω_M and Ω_H. In contrast to the conventional electromagnetism, the theory treats electric and magnetic fields in different ways and we have discussed each case separately.

It has been found that the cases of a positive and negative non-minimal coupling parameter λ are drastically different. For $\lambda < 0$, we identified physical finite-time singularities where the deceleration parameter diverges. In the electric case, the singularity comes from the modified evolution equation for the electric field, while in the magnetic case, it stems from the Einstein equations. In both cases, we identified a range of initial conditions (which

depend on λ) for which the singularities are inevitable. Although we have been unable to show the singularity is unavoidable during inflation, its mere existence in state space is a problem in itself and suggests that the theory is pathological for $\lambda < 0$, regardless of the other matter fields in the universe. Our result effectively rules out the possibility of generating anisotropy or magnetic field during inflation by employing this type of non-minimal coupling.

For $\lambda > 0$, it appears that the modification is relatively harmless as there is no singular behaviour and the Friedmann solution (F) is the future attractor for any perfect fluid satisfying $\gamma < 4/3$. For inflationary universes with $\gamma \ll 1$, we have found F to be stable, and consequently all anisotropies are washed out. In radiation and dust-dominated universes, however, we find a rather interesting phenomenology. Namely, in the electric case, the stability of F is time dependent and there is a transient period where both Ω_M and Ω_H grow. Both Ω_M and Ω_H reach their maximum values around the time $\Omega_M \sim \Omega_H$, and then Ω_M decays adiabatically but the decrease of Ω_H is much faster. Demanding the validity of the Maxwellian electromagnetism in terrestrial environments, this peak value has to occur at the latest by the time of big bang nucleosynthesis, which means the cosmological effect of Ω_H afterwards is always negligible. Therefore, all the cosmological constraints after nucleosynthesis are based on the dynamics of Ω_M and hence cannot be used to improve the upper bound on the non-minimal coupling constant λ. Nevertheless, a huge amplification of electric field energy may have taken place in the early universe ($\Omega_M \propto e^{8\alpha}$ when F is unstable). Even if electric fields are washed out during inflation ($\Omega_M \propto e^{-4\alpha}$ if $\gamma \simeq 0$), there could be enough time between reheating and big bang nucleosynthesis for Ω_M to grow significantly if inflation lasts not much more than 60 e-folds and the reheating occurs at a relatively high energy scale. After the amplification saturates, Ω_H decays quickly and soon becomes negligible so that it does not cause any significant problems at later times. We note that this amplification is characteristic of uniform electric fields; for uniform magnetic fields, the dynamics near F are practically identical to the conventional electromagnetic case.

Contrary to the minimally coupled Maxwell case, we found that the generalised energy-momentum tensor may source both a positive and negative shear. If the theory is merely taken to be some hypothetical vector-tensor theory (not electromagnetism), so that the Hubble normalised field energy may grow at late times, this opens up the possibility for cancellation effects in the CMB. In particular we saw that if Ω_H peaks around redshift $z \sim 0.6$, all CMB effects of the anisotropic expansion cancel exactly. The possibility of such a cancellation effect was mentioned in [86, 87] and to the best of our knowledge, Horndeski's theory is the first instance of a Lagrangian for which such a scenario may be dynamically realised.

In this work we have neglected all spatial derivatives, and focused on understanding the full non-linear nature of the system. Physically, uniform electric and magnetic fields correspond to a long wavelength limit of cosmic electric or magnetic fields. This approach has been fruitful as it enabled us to identify singularities at the background level and therefore rule out any interesting cosmological application of this theory for a negative non-minimal coupling. Although this makes the theory irrelevant in the context of in-

flation, we have demonstrated that a non-trivial phenomenology is possible in the early radiation-dominated universe. We point out that the short-wavelength limit appears to be an interesting direction of further investigation. Especially, when spatial derivatives are taken into account, it would be interesting to see if an amplification of cosmic magnetic fields could be possible before big bang nucleosynthesis. It is also necessary to check if the model avoids instabilities from inhomogeneous perturbations.

Acknowledgments

References

[1] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Modified Gravity and Cosmology*, arXiv:1106.2476.

[2] M. Milgrom and R. H. Sanders, *Rings and Shells of Dark Matter as MOND Artifacts*, *The Astrophysical Journal* **678** (May, 2008) 131–143, [arXiv:0709.2561].

[3] C. Brans and R. H. Dicke, *Mach's Principle and a Relativistic Theory of Gravitation*, *Physical Review* **124** (Nov., 1961) 925–935.

[4] K. Nordtvedt, *Post-Newtonian Metric for a General Class of Scalar-Tensor Gravitational Theories and Observational Consequences.*, *The Astrophysical Journal* **161** (Sept., 1970) 1059.

[5] R. V. Wagoner, *Scalar-Tensor Theory and Gravitational Waves*, *Physical Review D* **1** (June, 1970) 3209–3216.

[6] J. D. Barrow and K.-i. Maeda, *Extended inflationary universes*, *Nuclear Physics B* **341** (Sept., 1990) 294–308.

[7] T. V. Ruzmaikina and A. A. Ruzmaikin, *Gravitational Stability of an Expanding Universe in the Presence of a Magneric Field.*, *Soviet Astronomy* **14** (June, 1971) 963.

[8] J. D. Barrow and A. C. Ottewill, *The stability of general relativistic cosmological theory*, *Journal of Physics A: Mathematical and General* **16** (Aug., 1983) 2757–2776.

[9] J. D. Barrow and S. Cotsakis, *Inflation and the conformal structure of higher-order gravity theories*, *Physics Letters B* **214** (Dec., 1988) 515–518.

[10] T. P. Sotiriou, *f(R) theories of gravity*, *Reviews of Modern Physics* **82** (Mar., 2010) 451–497, [arXiv:0805.1726].

[11] T. Sotiriou, B. Li, and J. Barrow, *Generalizations of teleparallel gravity and local Lorentz symmetry*, *Physical Review D* **83** (May, 2011) 104030, [arXiv:1012.4039].

[12] A. Goldhaber and M. Nieto, *Terrestrial and Extraterrestrial Limits on The Photon Mass*, *Reviews of Modern Physics* **43** (July, 1971) 277–296.

[13] A. Barnes, *Cosmology of a charged universe*, *The Astrophysical Journal* **227** (Jan., 1979) 1.

[14] J. D. Barrow and R. R. Burman, *Particle physics and cosmology: New light on heavy light*, *Nature* **307** (Jan., 1984) 14–15.

[15] A. Dolgov and Y. Zeldovich, *Cosmology and elementary particles*, *Reviews of Modern Physics* **53** (Jan., 1981) 1–41.

[16] J. Webb, V. Flambaum, C. Churchill, M. Drinkwater, and J. Barrow, *Search for Time Variation of the Fine Structure Constant*, *Physical Review Letters* **82** (Feb., 1999) 884–887, [`9803165`].

[17] M. T. Murphy, J. K. Webb, and V. V. Flambaum, *Revision of VLT/UVES constraints on a varying fine-structure constant*, *Monthly Notices of the Royal Astronomical Society* **384** (Mar., 2008) 1053–1062, [`0612407`].

[18] J. Bekenstein, *Fine-structure constant: Is it really a constant?*, *Physical Review D* **25** (Mar., 1982) 1527–1539.

[19] H. v. Sandvik, J. Barrow, and J. a. Magueijo, *A Simple Cosmology with a Varying Fine Structure Constant*, *Physical Review Letters* **88** (Jan., 2002) 031302, [`0107512`].

[20] J. D. Barrow and D. F. Mota, *Qualitative analysis of universes with varying alpha*, *Classical and Quantum Gravity* **19** (Dec., 2002) 6197–6212, [`0207012`].

[21] J. D. Barrow, J. a. Magueijo, and H. v. B. Sandvik, *A cosmological tale of two varying constants*, *Physics Letters B* **541** (Aug., 2002) 201–210, [`0204357`].

[22] J. Barrow, H. v. Sandvik, and J. a. Magueijo, *Behavior of varying-alpha cosmologies*, *Physical Review D* **65** (Feb., 2002) 063504, [`0109414`].

[23] J.-P. Uzan, *The fundamental constants and their variation: observational and theoretical status*, *Reviews of Modern Physics* **75** (Apr., 2003) 403–455, [`0205340`].

[24] J. Barrow and S. Lip, *Generalized theory of varying alpha*, *Physical Review D* **85** (Jan., 2012) 023514, [`arXiv:1110.3120`].

[25] M. Turner and L. Widrow, *Inflation-produced, large-scale magnetic fields*, *Physical Review D* **37** (May, 1988) 2743–2754.

[26] B. Ratra, *Cosmological 'seed' magnetic field from inflation*, *The Astrophysical Journal* **391** (May, 1992) L1.

[27] E. Calzetta, A. Kandus, and F. Mazzitelli, *Primordial magnetic fields induced by cosmological particle creation*, *Physical Review D* **57** (June, 1998) 7139–7144, [`9707220`].

[28] M. Giovannini, *Magnetogenesis and the dynamics of internal dimensions*, *Physical Review D* **62** (Nov., 2000) 123505, [`0007163`].

[29] G. Lambiase and A. Prasanna, *Gauge invariant wave equations in curved space-times and primordial magnetic fields*, *Physical Review D* **70** (Sept., 2004) 063502, [`0407071`].

[30] K. E. Kunze, *Primordial magnetic seed fields from extra dimensions*, *Physics Letters B* **623** (Sept., 2005) 1–9, [`0506212`].

[31] K. Bamba and M. Sasaki, *Large-scale magnetic fields in the inflationary universe*, *Journal of Cosmology and Astroparticle Physics* **2007** (Feb., 2007) 030–030, [`0611701`].

[32] K. Kunze, *Primordial magnetic fields and nonlinear electrodynamics*, *Physical Review D* **77** (Jan., 2008) 023530, [`arXiv:0710.2435`].

[33] L. Campanelli, P. Cea, G. Fogli, and L. Tedesco, *Inflation-produced magnetic fields in RnF2*

and IF2 models, *Physical Review D* **77** (June, 2008) 123002, [arXiv:0802.2630].

[34] L. Campanelli, P. Cea, G. Fogli, and L. Tedesco, *Inflation-produced magnetic fields in nonlinear electrodynamics*, *Physical Review D* **77** (Feb., 2008) 043001, [arXiv:0710.2993].

[35] K. Bamba, N. Ohta, and S. Tsujikawa, *Generic estimates for magnetic fields generated during inflation including Dirac-Born-Infeld theories*, *Physical Review D* **78** (Aug., 2008) 043524, [arXiv:0805.3862].

[36] K. Bamba, C. Q. Geng, and S. H. Ho, *Large-scale magnetic fields from inflation due to ChernSimons-like effective interaction*, *Journal of Cosmology and Astroparticle Physics* **2008** (Nov., 2008) 013, [arXiv:0806.1856].

[37] L. Campanelli and P. Cea, *MaxwellKostecký electromagnetism and cosmic magnetization*, *Physics Letters B* **675** (May, 2009) 155–158, [arXiv:0812.3745].

[38] H. Mosquera Cuesta and G. Lambiase, *Primordial magnetic fields and gravitational baryogenesis in nonlinear electrodynamics*, *Physical Review D* **80** (July, 2009) 023013, [arXiv:0907.3678].

[39] L. Campanelli, P. Cea, and G. Fogli, *Lorentz symmetry violation and galactic magnetism*, *Physics Letters B* **680** (Sept., 2009) 125–128, [arXiv:0805.1851].

[40] K. E. Kunze, *Large scale magnetic fields from gravitationally coupled electrodynamics*, *Physical Review D* **81** (Feb., 2010) 043526, [arXiv:0911.1101].

[41] L. Ford, *Inflation driven by a vector field*, *Physical Review D* **40** (Aug., 1989) 967–972.

[42] W. Donnelly and T. Jacobson, *Coupling the inflaton to an expanding aether*, *Physical Review D* **82** (Sept., 2010) 064032, [arXiv:1007.2594].

[43] M. Gasperini, *Inflation and broken Lorentz symmetry in the very early universe*, *Physics Letters B* **163** (Nov., 1985) 84–86.

[44] S. Carroll and E. Lim, *Lorentz-violating vector fields slow the universe down*, *Physical Review D* **70** (Dec., 2004) 123525, [0407149].

[45] E. Lim, *Can we see Lorentz-violating vector fields in the CMB?*, *Physical Review D* **71** (Mar., 2005) 063504, [0407437].

[46] B. Li, D. Mota, and J. Barrow, *Detecting a Lorentz-violating field in cosmology*, *Physical Review D* **77** (Jan., 2008) 024032, [arXiv:0709.4581].

[47] J. Zuntz, P. Ferreira, and T. Zlosnik, *Constraining Lorentz Violation with Cosmology*, *Physical Review Letters* **101** (Dec., 2008) 261102, [arXiv:0808.1824].

[48] C. Armendariz-Picon, N. F. Sierra, and J. Garriga, *Primordial Perturbations in Einstein-Aether and BPSH Theories*, arXiv:1003.1283.

[49] T. Zlosnik, P. Ferreira, and G. Starkman, *Growth of structure in theories with a dynamical preferred frame*, *Physical Review D* **77** (Apr., 2008) 084010, [arXiv:0711.0520].

[50] X.-H. Meng and X.-L. Du, *A Specific Case of Generalized Einstein-aether Theories*, arXiv:1109.0823.

[51] M. Nakashima and T. Kobayashi, *CMB Polarization in Einstein-Aether Theory*, arXiv:1012.5348.

[52] T. Koivisto and D. F. Mota, *Vector field models of inflation and dark energy*, *Journal of Cosmology and Astroparticle Physics* **2008** (Aug., 2008) 021, [arXiv:0805.4229].

[53] A. Golovnev, V. Mukhanov, and V. Vanchurin, *Vector Inflation, Journal of Cosmology and Astroparticle Physics* **2008** (Feb., 2008) 6, [arXiv:0802.2068].

[54] K. Bamba and S. D. Odintsov, *Inflation and late-time cosmic acceleration in non-minimal Maxwell-$F(R)$ gravity and the generation of large-scale magnetic fields, Journal of Cosmology and Astroparticle Physics* **2008** (Jan., 2008) 20, [arXiv:0801.0954].

[55] J. Beltran Jimenez and A. L. Maroto, *A cosmic vector for dark energy, Phys.Rev.* **D78** (2008) 063005, [arXiv:0801.1486].

[56] T. S. Koivisto, D. F. Mota, and C. Pitrou, *Inflation from N-Forms and its stability, Journal of High Energy Physics* **2009** (Mar., 2009) 24, [arXiv:0903.4158].

[57] B. Himmetoglu, C. R. Contaldi, and M. Peloso, *Ghost instabilities of cosmological models with vector fields nonminimally coupled to the curvature, Physical Review D* **80** (Sept., 2009) 44, [arXiv:0909.3524].

[58] A. Golovnev, *Linear perturbations in vector inflation and stability issues, Physical Review D* **81** (Oct., 2009) 11, [arXiv:0910.0173].

[59] J. Barrow and J. Levin, *Chaos in the Einstein-Yang-Mills Equations, Physical Review Letters* **80** (Jan., 1998) 656–659, [9706065].

[60] Y. Jin and K.-i. Maeda, *Chaos of Yang-Mills field in class A Bianchi spacetimes, Physical Review D* **71** (Mar., 2005) 064007, [0412060].

[61] J. D. Barrow, Y. Jin, and K.-i. Maeda, *Cosmological Co-evolution of Yang-Mills Fields and Perfect Fluids*, 0509097.

[62] G. W. Horndeski, *Conservation of Charge and the Einstein-Maxwell Field Equations, J.Math.Phys.* **17** (1976) 1980–1987.

[63] G. Esposito-Farese, C. Pitrou, and J.-P. Uzan, *Vector theories in cosmology, Phys.Rev.* **D81** (2010) 063519, [arXiv:0912.0481].

[64] H. A. Buchdahl, *On a lagrangian for non-minimally coupled gravitational and electromagnetic fields, J.Phys.* **A12** (1979) 1037–1043.

[65] D. Lovelock, *The Einstein Tensor and Its Generalizations, Journal of Mathematical Physics* **12** (Mar., 1971) 498.

[66] D. Lovelock, *The Four-Dimensionality of Space and the Einstein Tensor, Journal of Mathematical Physics* **13** (June, 1972) 874.

[67] G. W. Horndeski, *Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space, International Journal of Theoretical Physics* **10** (Sept., 1974) 363–384.

[68] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Generalized G-inflation: Inflation with the most general second-order field equations, Prog.Theor.Phys.* **126** (2011) 511–529, [arXiv:1105.5723].

[69] A. De Felice, T. Kobayashi, and S. Tsujikawa, *Effective gravitational couplings for cosmological perturbations in the most general scalar-tensor theories with second-order field equations, Phys.Lett.* **B706** (2011) 123–133, [arXiv:1108.4242].

[70] A. De Felice and S. Tsujikawa, *Conditions for the cosmological viability of the most general scalar-tensor theories and their applications to extended Galileon dark energy models, JCAP* **1202** (2012) 007, [arXiv:1110.3878].

[71] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, *Self-tuning and the derivation of a class of scalar-tensor theories*, *Phys.Rev.* **D85** (2012) 104040, [arXiv:1112.4866].

[72] E. J. Copeland, A. Padilla, and P. M. Saffin, *The cosmology of the Fab-Four*, arXiv:1208.3373.

[73] S. A. Appleby, A. De Felice, and E. V. Linder, *Fab 5: Noncanonical Kinetic Gravity, Self Tuning, and Cosmic Acceleration*, *JCAP* **1210** (2012) 060, [arXiv:1208.4163].

[74] G. W. Horndeski and J. Wainwright, *Energy Momentum Tensor of the Electromagnetic Field*, *Phys.Rev.* **D16** (1977) 1691–1701.

[75] V. G. LeBlanc, *Asymptotic states of magnetic Bianchi I cosmologies*, *Class.Quant.Grav.* **14** (1997) 2281–2301.

[76] C. B. Collins, *Qualitative magnetic cosmology*, *Communications in Mathematical Physics* **27** (Mar., 1972) 37–43.

[77] C. Misner, K. Thorne, and J. Wheeler, *Gravitation*. W.H. Freeman, San Francisco, 1973.

[78] J. Wainwright and G. F. R. Ellis, *Dynamical Systems in Cosmology*. Cambridge University Press, 1997.

[79] Y. B. Zel'dovich, *The Hypothesis of Cosmological Magnetic Inhomogeneity.*, *Soviet Astronomy* **13** (Feb., 1970) 608.

[80] J. Barrow, *Cosmological limits on slightly skew stresses*, *Physical Review D* **55** (June, 1997) 7451–7460, [9701038].

[81] J. Barrow and R. Maartens, *Anisotropic stresses in inhomogeneous universes*, *Physical Review D* **59** (Dec., 1998) 043502, [9808268].

[82] J. Barrow, *Light elements and the isotropy of the Universe*, *Monthly Notices of the Royal Astronomical Society* **175** (May, 1976) 359–370.

[83] R. Lafrance and R. C. Myers, *Gravity's rainbow*, *Phys.Rev.* **D51** (1995) 2584–2590, [hep-th/9411018].

[84] J. D. Barrow, P. G. Ferreira, and J. Silk, *Constraints on a primordial magnetic field*, *Phys.Rev.Lett.* **78** (1997) 3610–3613, [astro-ph/9701063].

[85] M. Thorsrud, D. F. Mota, and S. Hervik, *Cosmology of a Scalar Field Coupled to Matter and an Isotropy-Violating Maxwell Field*, *JHEP* **1210** (2012) 066, [arXiv:1205.6261].

[86] T. Koivisto and D. F. Mota, *Anisotropic Dark Energy: Dynamics of Background and Perturbations*, *JCAP* **0806** (2008) 018, [arXiv:0801.3676].

[87] W. Lim, U. Nilsson, and J. Wainwright, *Anisotropic universes with isotropic cosmic microwave background radiation: Letter to the editor*, *Class.Quant.Grav.* **18** (2001) 5583–5590, [gr-qc/9912001].